Chris Piech
CS109
Lecture #3
Sept 28, 2018

Probability

It is that time in the quarter (it is still week one) when we get to talk about probability. Again we are going to build up from first principles. We will heavily use the counting that we learned earlier this week.

Event Space and Sample Space

Sample space, S, is set of all possible outcomes of an experiment. For example:

1. Coin flip: S = {Head, Tails}
2. Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
3. Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}
4. # emails in a day: $S = \{x \mid x \in \mathbf{Z}, x \geq 0\}$ (non-neg. ints)
5. YouTube hrs. in day: $S = \{x \mid x \in \mathbf{R}, 0 \leq x \leq 24\}$

Event Space, E, is some subset of S that we ascribe meaning to. In set notation ($E \subseteq S$).

1. Coin flip is heads: E = {Head}
2. ≥ 1 head on 2 coin flips: E = {(H, H), (H, T), (T, H)}
3. Roll of die is 3 or less: E = {1, 2, 3}
4. # emails in a day ≤ 20: $E = \{x \mid x \in \mathbf{Z}, 0 \leq x \leq 20\}$
5. Wasted day (≥ 5 YT hrs.): $E = \{x \mid x \in \mathbf{R}, 5 \leq x \leq 24\}$

Probability

In the 20th century humans figured out a way to precisely define what a probability is:

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

In English this reads: lets say you perform n trials of an experiment. The probability of a desired event E is the ratio of trials that result in E to the number of trials performed (in the limit as your number of trials approaches infinity).

That is mathematically rigorous. You can also apply other semantics to the concept of a probability. One common meaning ascribed is that P(E) is a measure of the chance of E occurring.

I often think of a probability in another way: I don't know everything about the world. So it goes. As a result I have to come up with a way of expressing my belief that E will happen given my limited knowledge. This interpretation acknowledges that there are two sources of probabilities: natural randomness and our own uncertainty.

Axioms of Probability

Here are some basic truths about probabilities:

Axiom 1: $0 \leq P(E) \leq 1$
Axiom 2: $P(S) = 1$
Axiom 3: $P(E^c) = 1 - P(E)$

You can convince yourself of the first axiom by thinking about the definition of probability. As you perform trials of an experiment it is not possible to get more events then trials (thus probabilities are less than 1) and its not possible to get less than 0 occurrences of the event.

The second axiom makes sense too. If your event space *is* the sample space, then each trial must produce the event. This is sort of like saying; the probability of you eating cake (event space) if you eat cake (sample space) is 1.

The third axiom comes from a deep philosophical point. Everything in the world must either be a potato or not a potato. Similarly, everything in the sample space must either be in the event space, or not in the event space.

Equally Likely Events

Some sample spaces have equally likely outcomes. We like those sample spaces, because there is a way to calculate probability questions about those sample spaces simply by counting. Here are a few examples where there are equally likely outcomes:

1. Coin flip: S = {Head, Tails}
2. Flipping two coins: S = {(H, H), (H, T), (T, H), (T, T)}
3. Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

Because every outcome is equally likely, and the probability of the sample space must be 1, we can prove that each outcome must have probability:

$$P(\text{Each outcome}) = \frac{1}{|S|}$$

If an event is a subset of a sample space with equally likely outcomes.

$$P(E) = \frac{\text{number of outcomes in E}}{\text{number of outcomes in S}} = \frac{|E|}{|S|}$$

Interestingly, this idea also applies to continuous sample spaces. Consider the sample space of all the outcomes of the computer function ``random" which produces a real valued number between 0 and 1, where all real valued numbers are equally likely. Now consider the event E that the number generated is in the range [0.3 to 0.7]. Since the sample space is equally likely, $P(E)$ is the ratio of the size of E to the size of S. In this case $P(E) = 0.4$.

When trying to solve a problem using equally likely sample spaces, you will use counting. How you set up your counting strategy for the sample space will determine if each outcome is equally likely. A nifty trick: make your objects distinct. Counting with distinct objects often makes the sample space events equally likely. Even if your objects are not distinct by default, you can make them distinct, as long as you do so in both the sample space and the event space.